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MODELING NETWORKS OF QUEUES

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Abstract. *The paper considers the modeling of networks of queues. Nodes in the network are modeled as memoryless queues. The focus is on open networks, since external packet arrivals and packet departures are permitted. Memoryless queues are characterized by a Poisson packet arrival process. This means that the interarrival times are exponentially distributed. The service times are characterized by an exponential distribution as well. Systems of queues could be used to model communication networks. Namely, after leaving one node in the network, there is a certain probability that a packet proceeds to another node in the network. Note that the external packet arrivals are also generated according to a Poisson process. In addition, there is a non-zero probability for the packets to leave the network. The network performance is illustrated by numerical examples.*

Keywords: *networks of queues; queues; modeling.*

1. INTRODUCTION

Computer networks are known as a system of interconnected computers for the intent of sharing digital information. Networking supports communication between two or more programs running on distant computers. In other words, a computer network is a collection of computers which are in some ways connected and exchange data between themselves [1–7]. There are some types of networks which are worth to mention:

- Local Area Networks (LANs) [8].
- Metropolitan Area Networks (MANs) [9].
- Wide Area Networks (WANs) [10].
- Wireless networks [11].

All these networks facilitate the transfer of data among computers. Each network has its own protocols and possibly incorporates different technologies. The routers or gateways interconnect different networks by packetizing the data in the format used by the particular network. The Internet has therefore emerged as a network of networks [12–15]. More recently, the Internet of Things (IoT) has appeared that has the ability to interconnect the world in an extraordinary way. The IoT has grown immensely over the years, and found applications in transportation, infrastructures, agriculture, healthcare, and manufacturing [16, 17]. Queueing theory has emerged as a viable alternative in modeling various facets of IoT networks [18–22].

2. POISSON PROCESS

A random process $\{A(t)|t \geq 0\}$ taking nonnegative integer values is a Poisson process with rate λ given that [23]
1. $A(t)$ is a counting process [24] that gives the number of arrivals that have taken place between 0 and time t, that is, for $A(0) = 0$ and for $s < t, A(t) - A(s)$ is the number of arrivals during the interval $(s, t]$.

2. The number of arrivals that take place during disjoint time intervals are independent.

3. The number of arrivals in any interval of length τ is characterized by the Poisson distribution with parameter $\lambda\tau$, that for all $t, \tau > 0$ and $n = 0,1, \ldots$ is given by

$$P[A(t+\tau) - A(t) = n] = e^{-\lambda\tau\frac{(\lambda\tau)^n}{n!}}\cdot \quad (1)$$

Given a Poisson process, we consider that the interarrival times are independent and exponentially distributed with parameter λ. The exponential probability density function is specified by $p(\tau) = \lambda e^{-\lambda\tau}$. It follows that, $P[\tau < s] = 1 - e^{-\lambda s}$ and $P[\tau > s] = e^{-\lambda s}$ for $s \geq 0$.

For $t \geq 0$ and $\delta \geq 0$, we have that the probability that there are no arrivals during the small interval δ is given by $P[A(t+\delta) - A(t) = 0] = 1 - \lambda\delta + o(\delta)$, the probability that there is one arrival during the small interval δ is given by $P[A(t+\delta) - A(t) = 1] = \lambda\delta + o(\delta)$, and the probability that there are two or more arrivals during the small interval δ is given by $P[A(t+\delta) - A(t) \geq 2] = o(\delta)$, where $o(\delta)$ is a function of δ such that $\lim_{\delta\to 0} \frac{o(\delta)}{\delta} = 0$.

If multiple independent Poisson processes $A_1, \dots, A_k$ are merged into a single process as a sum $A = A_1 + \cdots + A_k$, this process is also Poisson with a rate given by the sum of the rates of the component Poisson processes [23].

3. THE M/M/m QUEUE

The M/M/m queue has m servers [23]. A customer at the beginning of the queue is routed to any server that is available at that moment. The state transition diagram is shown in Figure 1.

We can write down the steady state probabilities p_k and take $\delta \to 0$, to have [23]

$$\lambda p_{k-1} = k\,\mu\,p_k, \qquad k \leq m, \tag{2}$$

$$\lambda p_{k-1} = m\,\mu\,p_k, \qquad k > m.$$

From these equations, we can obtain [23]

$$p_k = \begin{cases} p_0 \frac{(m\rho)^k}{k!}, & k \leq m, \\ p_0 \frac{m^m \rho^k}{m!}, & k > m, \end{cases} \tag{3}$$

where ρ, the utilization factor, is given by

$$\rho = \frac{\lambda}{m\mu} < 1. \tag{4}$$

From the equations above we can calculate p_0 with the condition, $\sum_{k=0}^{\infty} p_k = 1$, we have that

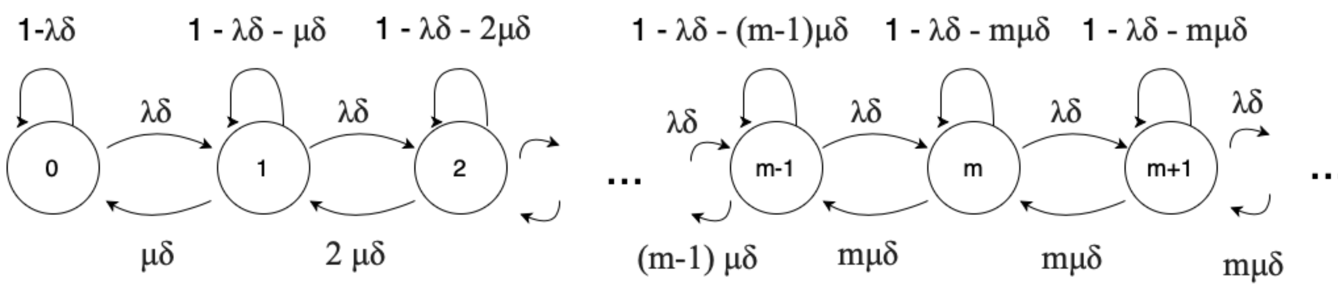

Figure 1. State transition diagram of the M/M/m queue.

$$p_0 = \left[1 + \sum_{k=1}^{m-1} \frac{(m\rho)^k}{k!} + \sum_{k=m}^{\infty} \frac{(m\rho)^k}{k!} \frac{1}{m^{k-m}}\right]^{-1} \tag{5}$$

and from there, we get [23]

$$p_0 = \frac{1}{\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!(1-\rho)}}. \tag{6}$$

4. MARKOVIAN QUEUEING NETWORKS

The focus is on a N node open Markovian network. The network is open in the sense that it permits external arrivals and departures. The external arrivals at the i^{th} node are generated by a Poisson source at an average rate of γ_i customers per second. The i^{th} node consists of a single queue with, say, m_i servers with an exponentially distributed service time. After the customer completes service at the i^{th} node it proceeds to the j^{th} node with probability r_{ij} where it represents an internal arrival to the j^{th} node. Note that it is possible for the customer to leave the network with a probability $1 - \sum_{j=1}^{N} r_{ij}$. The model also allows feedback to nodes that have been already visited. The total arrival rate at the i^{th} node which can be comprised of both external and internal arrivals is denoted by λ_i customers per second. It is therefore given by [23]

$$\lambda_i = \gamma_i + \sum_{j=1}^{N} \lambda_j r_{ji} \tag{7}$$

It follows as a result of Jackson's theorem that for such a network each node behaves as if the input is Poisson. This is so even though the arrival processes at different nodes are not necessarily Poisson. Therefore if $p(k_1, k_2, \ldots, k_N)$ denotes the equilibrium probability that there are k_i customers in the 1^{st} node, k_2 customers in the 2^{nd} node and so on, k_N customers in the N^{th} node then as a result of Jackson's theorem the joint probability can be represented as a product of the marginal probabilities $p_i(k_i)$ for $i = 1,2, \ldots, N$, that is [23]

$$p(k_1, k_2, \ldots, k_N) = p_1(k_1) p_2(k_2) \cdots p_N(k_N) \tag{8}$$

where $p_i(k_i)$ is the solution for the equilibrium probability of finding k_i customers in the queue of the i^{th} node as if it is an isolated queue operating by itself with an input arrival rate λ_i for $i = 1,2, \ldots, N$. In other words, the product form of the joint probability reveals the independence as indicated by the amazing result of the Jackson's theorem.

5. MODELING NETWORKS OF QUEUES

An example of an open Markovian network with $N = 4$ nodes is illustrated in Figure 2.

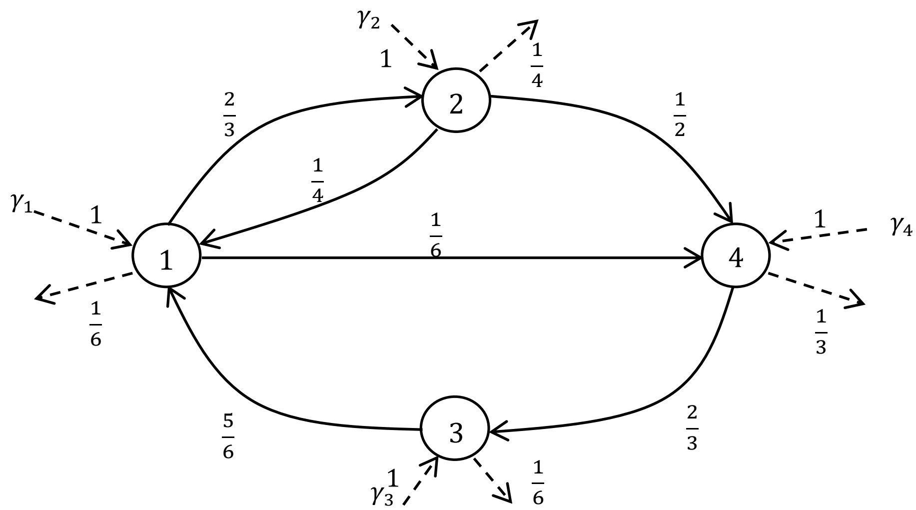

Figure 2. Example of an open network.

The branch labels are r_{ij}. These transition probabilities comprise the $N \times N$ matrix $\boldsymbol{R} = [r_{ij}]$. In the case of the example open network given in Figure 2, the matrix $\boldsymbol{R}$ is given by

$$\boldsymbol{R} = \begin{bmatrix} 0 & 2/3 & 0 & 1/6 \\ 1/4 & 0 & 0 & 1/2 \\ 5/6 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \end{bmatrix} \tag{9}$$

In order to determine the arrival rate to the i^{th} node λ_i, we need to solve $\lambda_i = \gamma_i + \sum_{j=1}^{N} \lambda_j r_{ji}$ for $i = 1,2, \ldots, N$. Let $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_N]$ and $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, \ldots, \gamma_N]$. The equation can then be rewritten as

$$\boldsymbol{\lambda} = \boldsymbol{\gamma} + \boldsymbol{\lambda R} \tag{10}$$

It follows that in order to find the arrival rates $\boldsymbol{\lambda}$, we need to evaluate

$$\boldsymbol{\lambda} = \boldsymbol{\gamma}(\boldsymbol{I} - \boldsymbol{R})^{-\mathbf{1}} \tag{11}$$

where $\boldsymbol{I}$ denotes the identity matrix given by

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Note that $\det(\boldsymbol{I} - \boldsymbol{R}) \neq 0$ for the matrix $(\boldsymbol{I} - \boldsymbol{R})$ to be invertible [25].

6. NUMERICAL EXAMPLES

The random input vector is binary, that is, $\gamma_i = 0$ or $\gamma_i = 1$ for $i = 1, \ldots, 4$.

The determinant of the matrix $\boldsymbol{I}-\boldsymbol{R}$ is

$$\det(\boldsymbol{I}-\boldsymbol{R}) = 0.5556 \tag{13}$$

Note that $\det(\boldsymbol{I}-\boldsymbol{R}) \neq 0$. Therefore the matrix $\boldsymbol{I}-\boldsymbol{R}$ is invertible and the inverse matrix $(\boldsymbol{I}-\boldsymbol{R})^{-1}$ is

$$(\boldsymbol{I}-\boldsymbol{R})^{-1} = \begin{bmatrix} 1.8 & 1.2 & 0.6 & 0.9 \\ 0.95 & 1.63 & 0.65 & 0.97 \\ 1.5 & 1 & 1.5 & 0.75 \\ 1 & 0.67 & 1 & 1.5 \end{bmatrix} \tag{14}$$

Therefore, the vector of arrival rates can be readily evaluated as $\boldsymbol{\lambda} = \boldsymbol{\gamma}(\boldsymbol{I}-\boldsymbol{R})^{-1}$**.** Note that the numerical example is implemented in MATLAB [26, 27].
The arrival rates for each of the nodes of the example open network are presented in Figure 3. It can be readily observed that for the considered time range the arrival rates are $\lambda_1 \lesssim 5.75$ packets per second at the 1^{st} node in the network, $\lambda_2 \lesssim 5$ packets per second at the 2^{nd} node in the network, $\lambda_3 \lesssim 4.25$ packets per second at the 3^{rd} node in the network, and $\lambda_4 \lesssim 3.5$ packets per second at the 4^{th} node in the network.

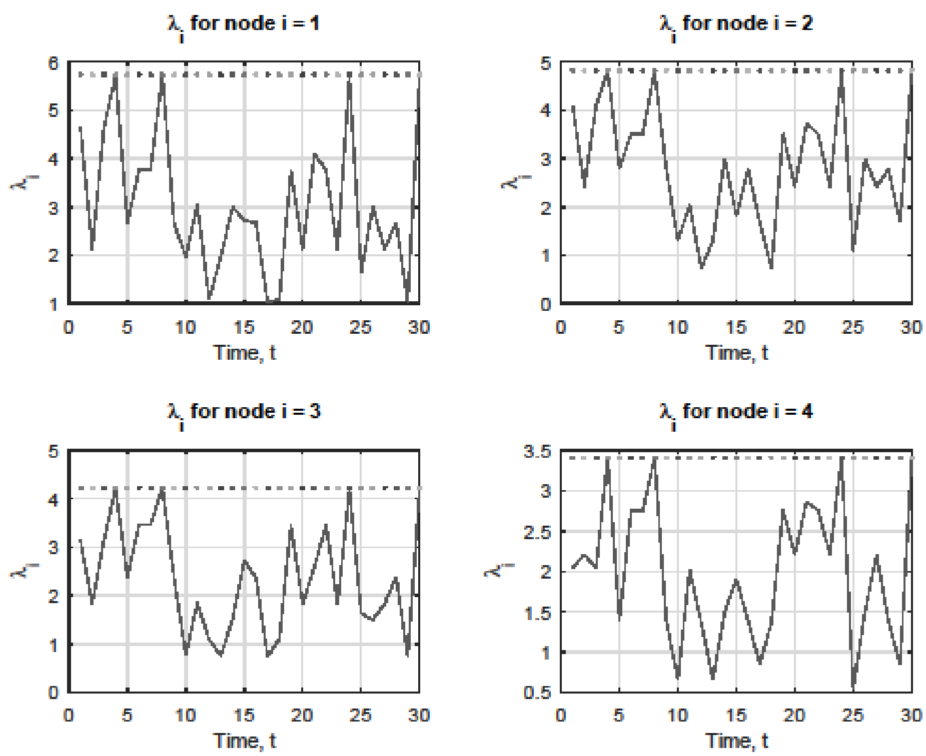

Figure 3. Arrival rates for each of the nodes in the open network.

7. CONCLUSION

The paper focused on the modeling of networks of queues with memoryless queues considered at each node in the network. The network was open in the sense that it permitted external arrivals and departures. The model was developed in MATLAB. It was illustrated through an example of a simple network where the network was modeled as a matrix. Arrival rates for all nodes in the network could then be obtained based on matrix calculations.

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