SOLVING TRIGONOMETRIC EQUATIONS IN PRIMARY SCHOOL?

Delčo Leškovski[1], Valentina Miovska[2]

[1]International Balkan University, Skopje, Republic of North Macedonia
[2]Faculty of Natural Sciences and Mathematics, Skopje, Republic of North Macedonia

Abstract. *Trigonometry is a subject taught at high school and university level. Since the basic trigonometric relations are first introduced in a right triangle, which is sufficiently studied in primary school, with an emphasis on the Pythagorean theorem, certain trigonometric problems can be solved by primary school students if they only know the definitions of the trigonometric functions in a right triangle as well as their values for characteristic acute angles (30°, 45° and 60°). We give a few examples of trigonometric equations that can be solved with basic knowledge of the trigonometric relations in a right triangle and using the geometric tools that the students in the upper grades of primary school already have. Everything is done in order to promote the creative teaching of mathematics, especially in the work with talented students, and to make the classes more interesting and fun. The objectives are to improve the math knowledge of the students, to develop their math skills and to motivate them to do independent research.*

Keywords: *trigonometry, trigonometric identities, trigonometric equations*

1. INTRODUCTION

Trigonometry is a subject taught at high school and university level. Few decades ago, trigonometric functions of an acute angle were taught in the eighth grade of primary education [1]. In recent years, this topic has been studied by the students in the second year of secondary education. Trigonometry studies the relationships involving side lengths and angles of triangles, so knowledge of geometry is very helpful in dealing with trigonometric problems. Since the basic trigonometric relations are first introduced in a right triangle, which is sufficiently studied in primary school, with an emphasis on the Pythagorean theorem, certain trigonometric problems can be solved by ninth-grade students if they only know the definitions of the trigonometric functions in a right triangle as well as their values for characteristic angles (30°, 45° and 60°). Many countries, such as Romania and North Korea, introduce basic trigonometry in middle school (grades 5-9). Vajiac in [6] concludes that basic trigonometric concepts are within reach of six graders. According to him, the schools may start with introducing elementary trigonometry in sixth-grade honors classes. It could be done with minimum effort and it would change the children attitude toward learning mathematics.

This idea of finding relationships between the angles and the sides in a right triangle is to motivate elementary school students to "discover" certain trigonometric identities on their own and then use them to solve some trigonometric equations. In the Macedonian educational system, the trigonometric equations are introduced to the students in the third year of secondary school. We give a few examples of trigonometric equations that can be solved with basic knowledge of the trigonometric relations in a right triangle and using the geometric tools that the students in ninth grade already have. Everything is done in order to promote the creative teaching of mathematics, especially in the work with talented students, and to make the classes more interesting and fun. The objectives are to improve the math knowledge of the students, to develop their math skills and to motivate them to do independent research.

At the beginning we present some results that we apply in our further work. Using elementary geometry and the definitions of the trigonometric functions in a right triangle and the unit circle, we compute the values of sine and cosine functions at $30^\circ, 45^\circ, 60^\circ$ and give proofs of some trigonometric identities. These results can be found in many books (see the references). Then we consider a few trigonometric equations and solve them for acute angles ($0 < x < 90^\circ$).

2. The Sine and Cosine Functions of $30^\circ, 45^\circ$ and 60°.

In this section we present some well-known results which can be found in many textbooks. Namely, the trigonometric ratios of $30^\circ, 45^\circ$ and 60° angles can be easily derived from $30^\circ-60^\circ-90^\circ$ and $45^\circ-45^\circ-90^\circ$ right triangles, "with the help" of the Pythagorean theorem.

Let ABC be an equilateral triangle with side length 1 and AD be the height of the triangle through the vertex A. Then ABD is right triangle and we have $\sin 60^\circ = \frac{\overline{AD}}{\overline{AB}} = \frac{\frac{\sqrt{3}}{2}\cdot 1}{1} = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{\overline{BD}}{\overline{AB}} = \frac{\frac{1}{2}\cdot 1}{1} = \frac{1}{2}$.

Since $\angle BAD = 30^\circ$, we have $\sin 30^\circ = \frac{\overline{BD}}{\overline{AB}} = \frac{\frac{1}{2}\cdot 1}{1} = \frac{1}{2}$ and $\cos 30^\circ = \frac{\overline{AD}}{\overline{AB}} = \frac{\frac{\sqrt{3}}{2}\cdot 1}{1} = \frac{\sqrt{3}}{2}$.

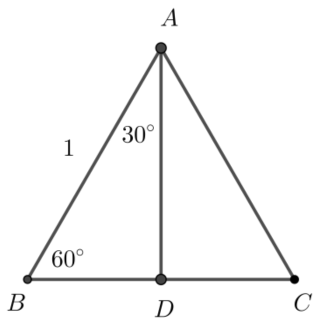

Figure 1

Let ABC be an isosceles right triangle with hypotenuse $\overline{AB}=1$. From the Pythagorean theorem we have $\overline{BC} = \overline{AC} = a = \frac{\sqrt{2}}{2}$. Then, $\sin 45^\circ = \frac{\overline{BC}}{\overline{AB}} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$ and $\cos 45^\circ = \frac{\overline{AC}}{\overline{AB}} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$.

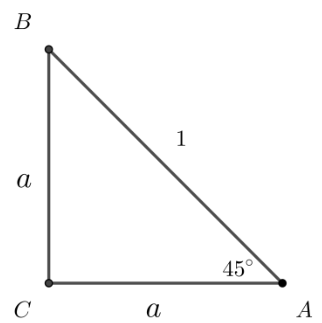

Figure 2

3. Some Basic Trigonometric Identities

Next, we give geometric proofs of some trigonometric identities which are used for the trigonometric equations in the following section. The geometric proofs in the trigonometry are not complete proofs since they depend on the figures, but are very suitable for the ninth-grade students. In the literature, one can find many examples of geometric proofs of various trigonometric identities [2, Jarrett] and very often those proofs are given as *proofs without words* [5, Nelsen].

3.1 Proof of $sin^2 x + cos^2 x = 1$

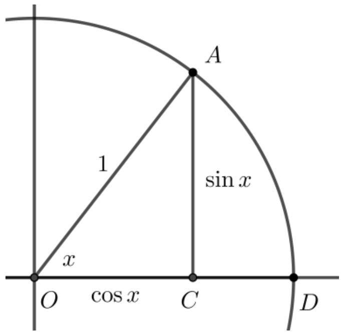

Figure 3

Let $\angle AOD = x$ and $C \in OD$, such that AC is perpendicular to OD. Then AOC is right triangle with the hypotenuse of length 1. We have $\sin x = \frac{\overline{AC}}{1}$, i.e., $\overline{AC} = \sin x$ and $\cos x = \frac{\overline{OC}}{1}$ i.e., $\overline{OC} = \cos x$. The Pythagorean theorem gives $\sin^2 x + \cos^2 x = 1$.

3.2 Proofs of $sin^2 \frac{x}{2} = \frac{1-cosx}{2}$ and $cos^2 \frac{x}{2} = \frac{1+cosx}{2}$

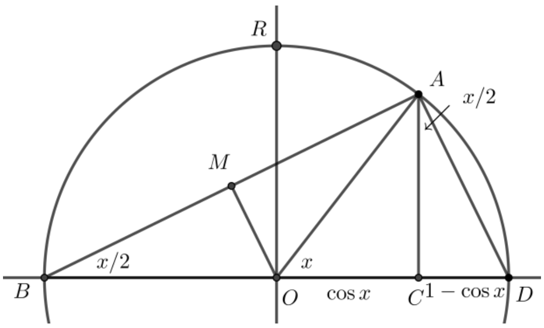

Figure 4

Let $\angle AOD = x$. Then $\angle ABD = \frac{x}{2}$ and $\angle BAD = 90°$. If M is the midpoint of the circle chord AB, then OM and AB are perpendicular to each other. From the right triangle OMB we obtain $\sin\frac{x}{2} = \frac{\overline{OM}}{1}$, i.e., $\overline{OM} = \sin\frac{x}{2}$ and

$\cos\frac{x}{2} = \frac{\overline{BM}}{1}$ i.e., $\overline{BM} = \cos\frac{x}{2}$. Since OM is a midsegment in the triangle ABD, it follows that $\overline{AD} = 2\sin\frac{x}{2}$, and since M is the midpoint of the segment AB, we have $\overline{AB} = 2\cos\frac{x}{2}$.

The triangles ABD and CAD are similar (both are right triangles and they have one common acute angle), so $\angle CAD = \frac{x}{2}$. Then, from the right triangle ACD we have $\sin\frac{x}{2} = \frac{\overline{CD}}{\overline{AD}} = \frac{1-\cos x}{2\sin\frac{x}{2}}$. Hence, $\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$.

From the right triangle ABC we have $\cos\frac{x}{2} = \frac{\overline{BC}}{\overline{AB}} = \frac{1+\cos x}{2\cos\frac{x}{2}}$. Hence, $\cos^2\frac{x}{2} = \frac{1+\cos x}{2}$.

4. Solving trigonometric equations

In this section, using elementary geometric tool, the definitions of trigonometric ratios and the results given above, we solve some trigonometric equations for acute angles ($0 < x < 90°$).

4.1) $\sin x = \cos\frac{x}{2}$, $0 < x < 90°$

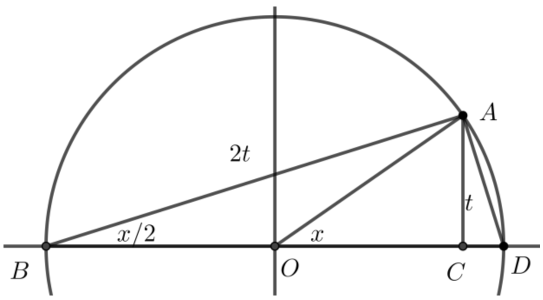

Figure 5

Let $\sin x = t$. From $\sin x = \cos\frac{x}{2}$, it follows that $\cos\frac{x}{2} = t$. From the right triangle ABC we obtain $\sin\frac{x}{2} = \frac{t}{2t} = \frac{1}{2}$. Hence, $\frac{x}{2} = 30°$, i.e., $x = 60°$.

4.2) $\cos x - \cos\frac{x}{2} + 1 = 0$, $0 < x < 90°$

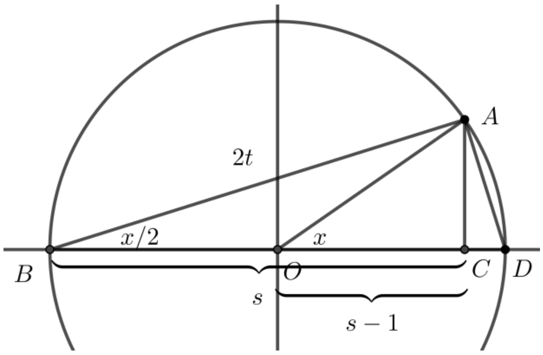

Figure 6

Let $\cos\frac{x}{2} = t$ and $1 + \cos x = s$. Then, $\cos x - \cos\frac{x}{2} + 1 = 0$ converts to $s - t = 0$. From the right triangles ABC and AOC, we obtain $\overline{AC}^2 = (2t)^2 - s^2$ and $\overline{AC}^2 = 1^2 - (1-s)^2$, respectively. Then, $(2t)^2 - s^2 = 1^2 - (1-s)^2$, $4t^2 - s^2 = 1 - 1 + 2s - s^2$, $4t^2 = 2s = 2t$. The last equation is equivalent to $t(2t-1) = 0$. Hence, $t = 0$ or $t = \frac{1}{2}$.

Since $t > 0$, it follows that $\cos\frac{x}{2} = \frac{1}{2}$. So $\frac{x}{2} = 60°$, i.e., $x = 120°$. Since we solve for $0 < x < 90°$, it follows that this equation has no solution.

4.3) $\sin x = \sqrt{3}\sin\frac{x}{2}$, $0 < x < 90°$

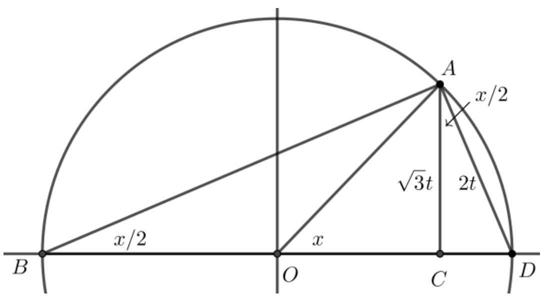

Figure 7

Let $\sin\frac{x}{2} = t$. Then $\overline{AD} = 2t$. From $\sin x = \sqrt{3}\sin\frac{x}{2}$ we obtain $\overline{AC} = \sin x = \sqrt{3}t$. It's clear that $\angle DAC = \frac{x}{2}$, so, from the right triangle ACD we have $\cos\frac{x}{2} = \frac{\sqrt{3}t}{2t} = \frac{\sqrt{3}}{2}$. Then, $\frac{x}{2} = 30°$, i.e., $x = 60°$.

4.4) $\sin\frac{x}{2}+\cos x=1$, $0<x<90°$

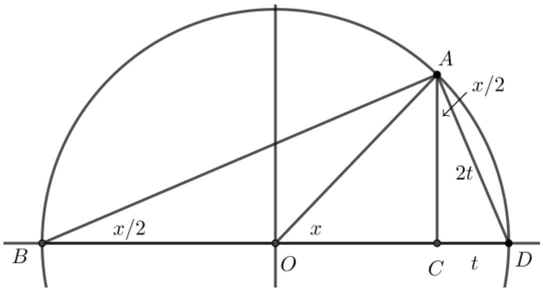

Figure 8

If $\sin\frac{x}{2}=t$ then $\overline{AD}=2t$. From $\sin\frac{x}{2}+\cos x=1$ we obtain $\overline{CD}=1-\cos x=t$. From the right triangle ACD we have $\sin\frac{x}{2}=\frac{t}{2t}=\frac{1}{2}$. Hence, $\frac{x}{2}=30°$, i.e., $x=60°$.

4.5) $2sin^2x-cosx, 0<c<90°$

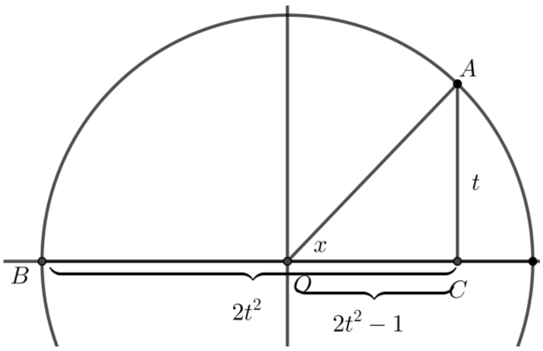

Figure 9

Let $\sin x=t$. Then $\overline{AC}=\sin x=t$ and $\overline{BC}=1+\cos x$. From $2\sin^2 x-\cos x=1$, it follows that $\overline{BC}=1+\cos x=2\sin^2 x=2t^2$. Hence, $\overline{OC}=\cos x=\overline{BC}-1=2t^2-1$. From the right triangle AOC we obtain $t^2+(2t^2-1)^2=1^2$, i.e., $4t^4-3t^2=0$. The last equation is equivalent to $t^2(4t^2-3)=0$. Hence, $t^2=0$ or $t^2=\frac{3}{4}$. Since $t>0$, it follows that $t=\frac{\sqrt{3}}{2}$, i.e., $\sin x=\frac{\sqrt{3}}{2}$. Finally, $x=60°$.

4.6) $\mathrm{tg}x = \sqrt{2}\sin x$, $0 < x < 90°$

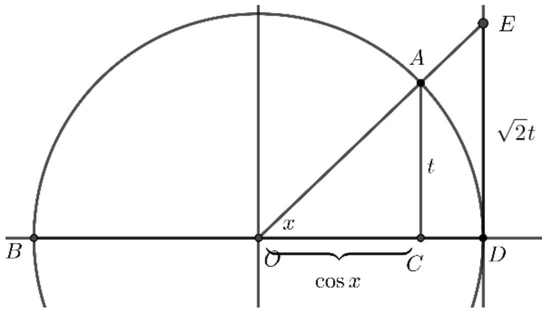

Figure 10

Let $\sin x = t$. From $\mathrm{tg}x = \sqrt{2}\sin x$ we have $\overline{ED} = \mathrm{tg}x = \sqrt{2}t$. Since the triangles AOC and EOD are similar, it follows that $\frac{\cos x}{1} = \frac{t}{\sqrt{2}t}$, i.e., $\cos x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Hence, $x = 45°$.

4.7) $\cos x = \sin\frac{x}{2}$, $0 < x < 90°$

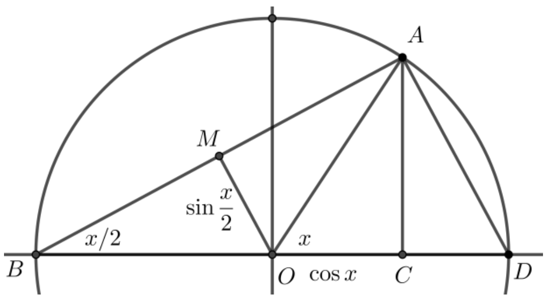

Figure 11

Let M be the midpoint of AB . Then, $\angle OMB = 90°$ and $\sin\frac{x}{2} = \frac{\overline{OM}}{\overline{OB}} = \frac{\overline{OM}}{1}$, so $\overline{OM} = \sin\frac{x}{2}$. From $\cos x = \sin\frac{x}{2}$ we have that $\overline{OM} = \overline{OC}$. It follows that the right triangles AMO and ACO are congruent (they have common hypotenuse and one equal leg). Therefore, $\angle AOM = \angle AOC = x$. Since $\angle AOM = \angle BOM$, it follows that $x + \frac{x}{2} = 90°$, i.e., $3x = 180°$. Hence, $x = 60°$.

4.8) $\sqrt{3}(1-cosx) = sinx,\ 0<x<90^\circ$

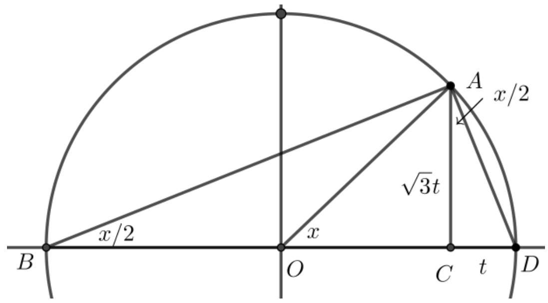

Figure 12

Let $\overline{CD} = 1-\cos x = t$. Then, the equation $\sqrt{3}(1-\cos x) = \sin x$ implies $\overline{AC} = \sqrt{3}t$. From the right triangle ACD we have $\mathrm{tg}\frac{x}{2} = \frac{t}{\sqrt{3}t} = \frac{1}{\sqrt{3}}$. Hence, $\frac{x}{2} = 30^\circ$, i.e., $x = 60^\circ$.

4.9) $3\mathrm{tg}^2 x = \frac{2}{\cos^2 x} + 1,\ 0<x<90^\circ$

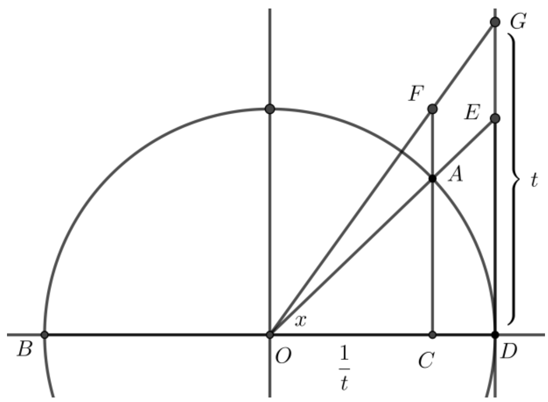

Figure 13

Let F be a point on the line AC, such that $\overline{FC} = 1$. Let OF intersects the tangent of the circle at D, at the point G . The triangles OCF and ODG are similar and therefore $\frac{\overline{FC}}{\overline{OC}} = \frac{\overline{GD}}{\overline{OD}}$, i.e. $\frac{1}{\cos x} = \frac{\overline{GD}}{1}$. Hence, $\overline{GD} = \frac{1}{\cos x}$.

Let $t = \frac{1}{\cos x}$. Then $\overline{OC} = \frac{1}{t}$ and $\sin x = \overline{AC} = \sqrt{1-\frac{1}{t^2}}$, i.e., $\sin^2 x = 1-\frac{1}{t^2}$ ... (1) The triangles AOC and EOD are similar and therefore $\frac{\overline{AC}}{\overline{OC}} = \frac{\overline{ED}}{\overline{OD}}$ i.e., $\frac{\sin x}{\frac{1}{t}} = \frac{\mathrm{tg}x}{1}$. Hence, $t = \frac{\mathrm{tg}x}{\sin x}$, i.e. $t^2 = \frac{\mathrm{tg}^2 x}{\sin^2 x}$ ... (2) From (1) and (2) we obtain

$tg^2x = t^2sin^2x = t^2\left(1-\frac{1}{t^2}\right) = t^2 - 1$... (3) From the initial equation $3tg^2x = \frac{2}{cos^2x} + 1$ and (3) we obtain $3(t^2-1) = 2t^2 + 1$, which is equivalent to $t^2 = 4$. Since $t > 0$, it follows that $t = 2$, i.e., $cosx = \frac{1}{2}$, Hence,$x = 60°$.

4.10) $2sin^2x + sinx - 1 = 0, 0 < x < 90°$

The given equation is equivalent to $2(1-\sin^2 x) = 1+\sin x$. Hence, $2\cos^2 x = 1+\sin x$ ... (1). From the right triangles AQP and RPA we have $\cos^2 x + (1+\sin x)^2 = b^2$ ... (2) and $a^2 + b^2 = 4$ ... (3), respectively. The triangles RPA and RAQ are similar. Therefore, $\frac{b}{2} = \frac{\cos x}{a}$, i.e., $a = \frac{2\cos x}{b}$ ... (4). Putting (4) in (3) we obtain $\frac{4\cos^2 x}{b^2} + b^2 = 4$. Hence, $\cos^2 x = \frac{4b^2 - b^4}{4}$ ... (5). From (1) and (2), it follows that $\cos^2 x + (2\cos^2 x)^2 = b^2$ … (6). If we put (5) in (6), we obtain $\frac{4b^2-b^4}{4} + (2\frac{4b^2-b^4}{4})^2 = b^2$. Hence, $(4b^2 - b^4)^2 = b^4$, i.e. $b^4(4-b^2)^2 = b^4$. Since $b^4 \neq 0$, it follows that $(4-b^2)^2 = 1$. Then $4-b^2 = 1$ or $4-b^2 = -1$, i.e., $b^2 = 3$ or $b^2 = 5$. If we substitute the last results in (5), we obtain $\cos^2 x = \frac{3}{4}$ or $\cos^2 x = \frac{-5}{4}$ (the second result is not possible). Since $0 < x < 90°$, it follows that $\cos x = \frac{\sqrt{3}}{2}$. Hence, $x = 30°$.

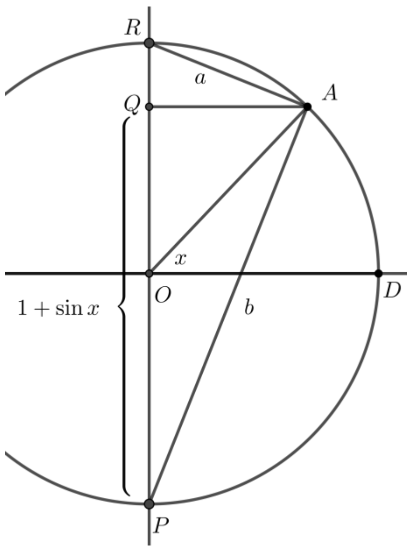

Figure 14

5. CONCLUSION

The purpose of this work is not to introduce the topic *Trigonometric equations* in regular classes which are realized according to a previously determined plan and program. The idea is to show that the elementary school students can derive trigonometric properties and solve some trigonometric equations with the knowledge they already have. They just need to be motivated to research and guided through the process. It could be done in honors classes or in the workshops for talented students. This approach makes the teaching mathematics more creative and interesting.

REFERENCES

1. Ansarov, R., Madevski, Ž., Celakoski, N.,1985, *Geometry (with collection of problems for 8th grade)*, Prosvetno Delo, Skopje (in Macedonian).
2. Jarrett, J. A., 1987, *A geometric proof of the sum-product identities for trigonometric functions*, Mathematics Teacher, 80(3), pp. 240-244.
3. Moore, K. C., 2012, *Coherence, quantitative reasoning, and the trigonometry of students*, Quantitative Reasoning and Mathematical Modeling: A Driver For STEM Integrated Education And Teaching In Context, 2, pp. 75–92.
4. Moore, K. C., LaForest, K. R., 2014, *The circle approach to trigonometry*, Mathematics Teacher, 107(8), pp. 616-623.
5. Nelsen, B. R., 1993, *Proofs without words (Exercises in visual thinking)*, Mathematical association of America.
6. Vajiac, B., Snow, J., 2009, *Mathematical roots: Can a sixth grader do trig?*, Mathematics Teaching In The Middle School, 15(3), pp. 176-180