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# Some New Inequalities for Harmonic Mathieu Series

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**ABSTRACT**
In this paper we derive some new estimates for some class of harmonic Mathieu series, using their integral representations and well known results for the harmonic numbers and the polygamma functions.

**Keywords:** ***harmonic Mathieu series, psi function, polygamma functions, inequalities***

## INTRODUCTION

The classical Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for k ∈ N are called the polygamma functions.

The harmonic numbers $H_n$ are defined and represented analytically by

$$H_0 = 0 \qquad \text{and} \qquad H_n = \sum_{k=1}^{n} \frac{1}{k} = \psi(n+1) + \gamma, \tag{2}$$

where $\gamma$ is the celebrated Euler-Mascheroni constant given by

$$\gamma = \lim_{n\to\infty} (H_n - \ln n) = \lim_{n\to\infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \approx 0.57721560... \tag{3}$$

For the harmonic numbers, the following identities hold true:

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n \tag{4}$$

$$\sum_{k=1}^{n} (-1)^{k-1} H_k = \frac{\left(1-(-1)^n\right)H_n - H_{[n/2]}}{2} \tag{5}$$

**MAIN RESULTS**

The aim of this paper is to establish new inequalities for some harmonic Mathieu series.

For our main results, the following lemmas are needed:

***Lemma 1*** ([1]). The following result holds true:

$$\int_0^u \frac{x^{\mu-1}}{(1+\beta x)^{\upsilon}}\,dx = \frac{u^{\mu}}{\mu}\cdot {}_2F_1(\upsilon,\mu;\mu+1;-\beta u) \qquad \left(\operatorname{Re}\mu>0, \left|\arg(1+\beta u)\right|<\pi\right). \tag{6}$$

where

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n\cdot(\beta)_n}{(\gamma)_n}\cdot\frac{z^n}{n!} \tag{7}$$

is the Gauss hypergeometric function and

$$(\alpha)_0 = 1;\ (\alpha)_n = \alpha\cdot(\alpha+1)\cdot\ldots\cdot(\alpha+n-1) \tag{8}$$

is the Pochhammer symbol.

The series in (7) is convergent for $|z|<1$.

In particular, for $\upsilon=1$ we have

$$\int_0^u \frac{x^{\mu-1}}{1+\beta x}\,dx = \frac{u^{\mu}}{\mu}\cdot {}_2F_1(1,\mu;\mu+1;-\beta u) \qquad \left(\operatorname{Re}\mu>0, \left|\arg(1+\beta u)\right|<\pi\right), \tag{9}$$

***Lemma 2*** ([2]). For $x > 0$ and $k \in N$, we have

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \tag{10}$$

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1}\psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}. \tag{11}$$

In particular, for $k = 1$, (11) becomes

$$\frac{1}{x} + \frac{1}{2x^2} < \psi'(x) < \frac{1}{x} + \frac{1}{x^2}. \tag{12}$$

***Lemma 3*** ([4]). The following double inequality holds true

$$\frac{1}{2(n+1)} < H_n - \ln n - \gamma < \frac{1}{2n} \tag{13}$$

***Lemma 4*** ([3]). The following integral representation holds true:

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+x)^{\mu}} = \mu \int_0^{\infty} \left( \int_0^{[t]} \frac{\psi(1+u) + (u-[u])\psi'(1+u) + \gamma}{(t+x)^{\mu+1}} du \right) dt \qquad (\mu > 1) \tag{14}$$

***Lemma 5*** ([3]). The following integral representation holds true:

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+x)^{\mu}} = \mu \int_1^{\infty} \frac{([t]+1)H_{[t]} - [t]}{(t+x)^{\mu+1}} dt \qquad (\mu > 1) \tag{15}$$

***Lemma 6*** ([3]). The following integral representation holds true:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{(H_n+x)^p} = \int_0^{\infty} \frac{(pu-u-x)\cos^2\left(\frac{\pi}{2}\left[\psi^{-1}(u-\gamma)\right]\right)}{(x+u)^{p+1}} du \qquad (x > 0,\, p \in \square \setminus \{1\}) \tag{16}$$

***Remark 1.*** For our results we use the right-hand sides in (10), (12) and (13).

***Remark 2.*** For our results we also use the set of basic inequalities

$$\ln(1+x) < x \ (x>0)\,;\ x-[x]<1\,;\ x-1<[x]<x\,. \tag{17}$$

Applying the inequalities (10), (12) and (17) in the integral representation (14), doing straightforward calculation and using the integral formula (9), we obtain the following result:

***Theorem 1***. For $\mu > 2$ and $x > \frac{1}{2}$, the following estimate holds true:

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+x)^{\mu}} \le \frac{\mu}{2(\mu-2)x^{\mu-2}} + \frac{\mu(2\gamma+1-2x)}{2(\mu-1)x^{\mu-1}} + \frac{x^2-(2\gamma+1)x+2}{2x^{\mu}} - \frac{\mu}{(\mu+1)x^{\mu+1}}\ {}_2F_1\left(1,\mu+1;\mu+2;\frac{x-1}{x}\right) \tag{18}$$

Applying the inequalities (13) and (17) in the integral representation (15), doing straightforward calculation and using the integral formula (9), we obtain the following result:

***Theorem 2***. For $\mu > 2$ and $x > 0$, the following estimate holds true:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{(n+x)^{\mu}} \le{}& \frac{1}{(1+x)^{\mu}} + \frac{x^2+(1-\gamma)x+\gamma-\frac{1}{2}}{(2+x)^{\mu}} + \frac{\mu}{(\mu-2)(2+x)^{\mu-2}} - \frac{\mu(2x+1-\gamma)}{(\mu-1)(2+x)^{\mu-1}} \\ &+ \frac{\mu}{(\mu+1)(2+x)^{\mu+1}}\ {}_2F_1\left(1,\mu+1;\mu+2;\frac{x+1}{x+2}\right)\end{aligned} \tag{19}$$

From the integral representation (16), using $|\cos x| \le 1$ and straightforward calculation, we obtain the following, not sharp but elegant inequality:

***Theorem 3.*** For $p \in \square \setminus \{1\}$, the following estimate holds true:

$$\left|\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{(H_n+x)^{p}}\right| \le \frac{2}{x^{p-1}}\left(1-\left(\frac{p-1}{p}\right)^{p}\right) \qquad (x>0) \tag{20}$$

## REFERENCES

[1] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products* (Corrected and Enlarged Edition prepared by A. Jeffrey and D. Zwillinger), Sixth ed., Academic Press, New York (2000)

[2] B.-N. Guo and F.Qi, *Two new proofs of the complete monotonicity of a function involving the psi function*, Bull. Korean Math. Soc. 47 (2010), no. 1, 103-111

[3] H.M. Srivastava, Ž. Tomovski and D. Leškovski, *Some families of Mathieu type series and Hurwitz-Lerch zeta functions and associated probability distributions*, Appl. Comput. Math., V.14, N.3, Special Issue (2015), 349-380

[4] R. M. Young, *Euler's constant*, Math. Gaz. 75 (1991), 187-190